$V^{(\varepsilon)}$ and V_τ of the second player, and the strategies V_τ and V^* of the second player replaced by the strategies U_τ and U^* of the first player. This is equally applicable to Theorem 3.2.

BIBLIOGRAPHY

1. Fleming, W. H., The convergence problem for differential games. J. Math. Anal. and Appl., Vol. 3, 1961.
2. Pontriagin, L. S., On linear differential games. 2. Dokl. Akad. Nauk SSSR, Vol. 175, №4, 1967.
3. Smol'iakov, E. R., Differential games in mixed strategies. Dokl. Akad. Nauk SSSR, Vol. 191, №1, 1970.
4. Varaiya, P. and Lin, J., Existence of saddle points in differential games. SIAM J. Control Vol. 7, №1, 1969.
5. Krasovskii, N. N., On a problem of tracking. PMM Vol. 27, №2, 1963.
6. Krasovskii, N. N. and subbotin, A. I., On the structure of differential games. Dokl. Akad. Nauk SSSR, Vol. 190, №3, 1970.
7. Krasovskii, N. N., On a differential convergence game. Dokl. Akad. Nauk SSSR, Vol. 193, №2, 1970.
8. Halmos, P. R., Measure Theory. Van Nostrand, 1950.
9. Liusternik, L. A. and Sobolev, V. N., Elements of Functional Analysis. M. "Nauka", 1965.
10. Filippov, A. F., Differential equations with a discontinuous right-hand side. Matem. sb., Vol. 51(93), №1, 1960.
11. Pshenichnyi, B. N., The Necessary Conditions of Extremum. M., "Nauka", 1969.

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THE ACCURACY OF CERTAIN NONLINEAR CONTROL SYSTEMS WITH RESTRICTIONS AND LAG

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L. S. GNOENSKII
(Moscow)

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The accuracy with which a nonlinear control system with lag reproduces an arbitrary action belonging to a certain class of functions is examined. The maximum errors arising in reproducing the action and their dependence on the parameters of the controlled object, and on the law of control used, are estimated.

1. Statement of the problem. Consider a closed system consisting of a controlled object and a regulator. The purpose of this system is to reproduce, using the initial value of the object $y(t)$, a previously unknown controlling action $x(t)$ whose rate of change

$$x'(t) \equiv \varphi(t), \quad |\varphi(t)| \leqslant m, \quad x(0) = 0 \tag{1.1}$$

is bounded, belonging to the class of functions F. The quality of performance of the system, which is at rest at $t \leqslant 0$, will be characterized by the maximum error

$$\varepsilon_{\max}(t) = \max|\varepsilon(t)|, \quad \varepsilon(t) = x(t) - y(t) \qquad (x \in F) \tag{1.2}$$

and the quantity

$$\varepsilon_\infty = \lim \varepsilon_{\max}(t) \qquad (t \to \infty)$$

This is therefore essentially the same problem on accumulation of perturbations investigated in [1, 2].

The behavior of the controlled object is described by the following differential equations:

$$c_0 y^{(n)}(t) + \ldots + c_{n-2} y''(t) + y'(t) = u(t-\tau) \tag{1.3}$$

$$y(0) = \cdots = y^{(n-1)}(0) = 0 \quad u(t_1) = 0 \quad (-\tau \leqslant t_1 \leqslant 0) \tag{1.4}$$

The controlling signal $u(t)$ is modulo bounded by the constant u_0. Direct feedback

$$u(t) = k\varepsilon(t)\left(|\varepsilon(t)| \leqslant \frac{u_0}{k}\right), \quad u(t) = u_0 \operatorname{sign} k\varepsilon(t) \quad \left(|\varepsilon(t)| > \frac{u_0}{k}\right) \tag{1.5}$$

is used as the law of control realized by the regulator.

Equations (1.3) and (1.5) describe the behavior of a closed, astatic servomechanism with lag and a bounded nonlinearity. The structural scheme of this servomechanism is shown in Fig. 1, where $F_1(p)$ is the transfer function of the object

$$F_1(p) = \frac{e^{-p\tau}}{Q(p)} = \frac{Y(p)}{U(p)}, \quad Q = pL, \quad L(p) = c_0 p^{n-1} + \cdots + c_{n-2} p + 1 \tag{1.6}$$

Functions $Y(p)$ and $U(p)$ are Laplace transforms of $y(t)$ and $u(t)$.

The upper bounds for ε_∞ are given below, showing the dependence of these estimates on the parameters u_0, m, τ, k and on the distribution of zeros of the polynomial $L(p)$, this distribution being assumed to be known. This assumption is justifiable, since an open system often contains a series of elements joined in a sequence and described by low, first- or second-order equations.

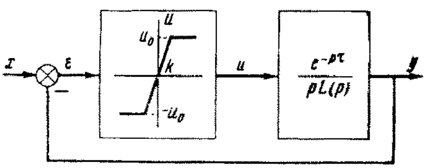

Fig. 1

The present problem is one of pursuit, in which the distance between the pursued and the pursuer is defined by the function $\varepsilon(t)$. The quantity $\varepsilon_{\max}(t)$ is not zero even for $u_0 > m$ and the initial conditions (1.1) and (1.4) hold, since the pursued object is "inertialess", i.e. it can attain its maximum velocity m instantaneously, while the pursuing object has inertial properties determined by the distribution of zeros in the polynomial $L(p)$. The law of control (1.5) chosen is not optimal. It is however widely used in practice for its simplicity of application, since the information of higher derivatives which is difficult to obtain, is not required. Nevertheless, ε_∞ can be sufficiently small, provided that the controlled object has desired dynamic properties.

From the stationary state of the system it follows that $\varepsilon_{\max}(t) < \varepsilon_\infty$. If $\varepsilon_\infty < u_0 k^{-1}$, then by (1.5) the closed system behaves as a linear one and can be described by the equation

$$c_0 y^{(n)}(t) + \ldots + c_{n-2} y''(t) + y'(t) + ky(t-\tau) = kx(t-\tau) \tag{1.7}$$

$$y(t_1) = x(t_1) = 0 \quad (-\tau \leqslant t_1 \leqslant 0), \qquad y(0) = \cdots = y^{(n-1)}(0) = 0$$

Theorem 1.3 gives the estimate for $\varepsilon_{\max}(t)$ and ε_∞ in a linear system. If this estimate exceeds $u_0 k^{-1}$, it cannot be guaranteed that the system satisfies (1.7), and in this case Theorem 1.1 must be used, the latter containing an estimate for ε_∞ in the nonlinear system (1.3) and (1.5).

The estimate of $\varepsilon_{max}(t)$ for linear systems without lag is given in [3, 4]. Estimates of $\varepsilon_{max}(t)$ for linear systems with lag were given by the author in his communication at the Second All-Union Conference on the Theory of Equations with Deviating Argument.

Let us assume that all zeros p_j of the polynomial $L(p)$ arranged in order of decreasing real parts satisfy the conditions

$$p_j = -\alpha_j(1 + i\mu_j), \qquad \alpha_j \geqslant a_0 > 0 \qquad (j = 1, \ldots n-1) \tag{1.8}$$
$$\alpha_1 = a_0, \quad \max|\mu_j| = \mu_0$$

In the theory of automatic control the quantities a_0 and μ_0 denote the degree of stability and the oscillation of $L(p)$, respectively.

Theorem 1.1. In a nonlinear system with lag (1.3) and (1.5) the maximum error ε_∞ does not exceed

$$\varepsilon_\infty < G_0 = u_0 \left\{k^{-1} + (a_0\gamma)^{-1}\left[D_0 + (2\tau a_0\gamma + 1)D_1 - 1 + mu_0^{-1} - \left(1 - \frac{m}{u_0}\right)\ln\frac{D_0 + 2D_1e^{a_0\gamma\tau} - D_1}{1 - mu_0^{-1}}\right]\right\} \qquad (m < u_0) \tag{1.9}$$

$$\varepsilon_\infty < G_0 = u_0\{k^{-1} + (a_0\gamma)^{-1}[D_0 + (2\tau a_0\gamma + 1)D_1]\} \quad (m = u_0)$$
$$\varepsilon_\infty = \infty \quad (m > u_0)$$

Here

$$D_1 = D_2(a_0(1-\gamma)(\alpha_2 - a_0\gamma)^{-1})^{\varkappa} \quad (\alpha_2 \neq a_0), \qquad D_1 = D_2e^{-1} \quad (\alpha_2 = a_0) \tag{1.10}$$

$$D_2 = \frac{1-\gamma}{\gamma}\prod_{j=1}^{n-1}\frac{\sqrt{1+\mu_j^2}}{\beta_j}, \qquad \beta_j = \frac{\alpha_j - a_0\gamma}{\alpha_j}, \qquad \varkappa = \frac{a_0(1-\gamma)}{\alpha_2 - a_0}$$

The following alternatives are possible: (1) at least one real zero of $L(p)$ is present on the line

$$\operatorname{Re} p = -a_0 \tag{1.11}$$

and (2) no real zeros are present on the line (1.11). The correspnding values of the relevant quantities are:

$$(1)\ D_0 = \frac{1}{2\gamma}\prod_{j=2}^{n-1}\nu_j, \qquad (2)\ D_0 = \frac{r}{2\gamma}\prod_{j=3}^{n-1}\nu_j \quad (n>3) \tag{1.12}$$
$$(1)\ D_0 = \frac{\nu_2}{2\gamma}, \quad (2)\ D_0 = \frac{r}{2\gamma} \quad (n=3), \qquad D_0 = D_1 = \gamma = 1 \quad (n=2)$$

$$\nu_j = \left(\frac{1+\mu_j^2}{2\mu_j\beta_j}\right)^{1/2} \quad (|\mu_j| \geqslant \beta_j), \qquad \nu_j = \left(\frac{1+\mu_j^2}{\mu_j^2+\beta_j^2}\right)^{1/2} \quad (|\mu_j| < \beta_j) \tag{1.13}$$

$$r = (1+\mu_1^2)\ \{2\ [(1-\gamma)^2 - \gamma^2 - \mu_1^2 + ((1-2\gamma)^2 + 2\mu_1^2((1-\gamma)^2 + \gamma^2) + \mu_1^4)^{1/2}]\}^{-1/2}, \quad 0 < \gamma \leqslant (1+\sqrt{2})^{-1}$$

where γ is an arbitrary number.

Note. Estimates of ε_∞ in the first order ($n = 1$) equation can be obtained from the estimates for $n = 2$ by going to the limit $a_0 \to \infty$

$$\varepsilon_\infty < u_0 [k^{-1} + \tau (1 + m u_0^{-1})] \quad (m \leqslant u_0), \qquad \varepsilon_\infty = \infty \quad (m > u_0) \qquad (1.14)$$

Proof of Theorem 1.1 is given in Sect. 2. From (1.9) and (1.13) it follows that when the degree of stability a_0 of $L(p)$ is large, the oscillation μ_0 is limited and the lag τ is small, the maximum error ε_∞ can be made small by choosing a sufficiently large k. The quantities a_0, μ_0, τ define the dynamic properties of the object.

Let us now consider the estimation of $\varepsilon_{max}(t)$ in the closed linear system (1.7). The quantity $\varepsilon_{max}(t)$ depends on the degree of stability δ^* of the quasipolynomial $N(p)$ corresponding to Eq. (1.7)

$$N(p) = Q(p) + \varkappa e^{-p\tau}, \qquad \delta^* = \min_j (-\operatorname{Re} p_j^*), \qquad N(p_j^*) = 0 \quad (1.15)$$

Let us find the least degree of stability of $N(p)$ that can be attained for the chosen value of the amplification factor k.

T h e o r e m 1 . 2. Let the amplification factor k satisfy the condition

$$k = \lambda\, |Q(-\delta)| \exp(-\delta\tau), \qquad (1 \leqslant \lambda \leqslant \lambda_1) \qquad (1.16)$$

Then the degree of stability δ^* of the quasipolynomial $N(p)$ is greater than

$$\delta = \pi\delta_1 \left(2\delta_1\gamma_1(0)\sqrt{\lambda_1^2 W^2 - 1} + \pi\right)^{-1} \qquad (1.17)$$

where δ_1 is the least root of the quadratic equation

$$z\gamma_1(z) = 1, \quad \gamma_1(z) = \tau_1 + \frac{n-1}{a_0 - z}, \quad \tau_1 = \frac{\pi\tau\lambda_1}{2} + \frac{\lambda_0}{a_0}, \quad W^2 = \prod_{j=1}^{n-1} A_j \qquad (1.18)$$

$$A_j = 1 \quad (|\mu_j| \leqslant y_j), \quad A_j = \frac{\mu_j^2 + y_j^2}{2\mu_j y_j} \quad (|\mu_j| > y_j), \qquad y_j = 1 - \frac{\delta_1}{\alpha_j}$$

$\lambda_0 \geqslant 0$ and $\lambda_1 \geqslant 1$ are arbitrary numbers.

C o r o l l a r y . Function $\gamma_1(z)$ in (1.18) can be replaced by

$$\gamma_1(z) = \tau_1 + \frac{q}{a_0 - z} + \sum_j \frac{1}{\alpha_j - a_0} \qquad (1.19)$$

where q is the number of zeros of $L(p)$ lying on the line (1.11) and the sum is taken over all remaining zeros of $L(p)$.

When $L(p)$ has zeros at a large distance from the line (1.11), the estimate based on the corollary may be found to be more accurate.

The proof of Theorem 1.2 is given in Sect. 3. The condition that $\lambda \in [1, \lambda_1]$ is also justified by technical considerations, since it is difficult to maintain k at the required value with sufficient accuracy.

The greatest attainable degree of stability δ^*_{max} of $N(p)$ is easily defined for a first degree equation at any value of k. Let us compare this value with the estimate obtained from Theorem 1.2 for $\lambda_1 = 1$

$$\delta^*_{max} = \tau^{-1} \quad (k = \tau^{-1}e^{-1}), \qquad \delta = 2\pi^{-1}\tau^{-1} \quad (k = 2\pi^{-1}\tau^{-1}e^{-2/\pi})$$

Approximate methods for investigating the distribution of zeros of the characteristic polynomial of a closed system relative to the amplification factor k, using the root hodograph were employed in e. g. [5]. For systems with lag similar methods were used in [6]. Upper bounds of the degree of stability attainable are given in [7, 8].

Using Theorem 1.2 we shall give the estimate for $\varepsilon_{max}(t)$ in the linear system (1.7).

T h e o r e m 1 . 3. If the amplification factor k satisfies the condition

$$k = \lambda |Q(-\delta)| e^{-\delta\tau} \qquad (1 < \lambda_2 < \lambda < \lambda_3 < \lambda_1) \tag{1.20}$$

then we have, in the closed linear system (1.7),

$$\varepsilon_{\max}(t) < m (\pi\delta)^{-1} G_1 (1 - e^{-\delta t}), \qquad \varepsilon_\infty < m (\pi\delta)^{-1} G_1 \tag{1.21}$$

Here δ is defined in Theorem 1.2,

$$G_1 = \frac{\ln (h + (1 + h^2)^{1/2})}{R_0} + \frac{(1 + h^2)^{1/2} (\pi / 2 - \operatorname{arc\,tg} h)}{\sqrt{(h^2 s + 1)(\lambda_3 W)^{-2}} - 1}$$
$$h = \frac{\arccos (\delta\gamma_1(\delta))}{\delta\gamma_1(\delta)}, \qquad s = 1 + \sum^* \frac{\delta^2}{\alpha_j^2} \tag{1.22}$$

The sum in (1.22) is computed over all real zeros of $L(p)$

$$R_0 = (1 - \lambda_2^{-1}) \quad (D_3 < 1), \qquad R_0 = (1 - \lambda_2^{-1})(2D_3^{-1} - D_3^{-2})$$
$$D_3 = \frac{\pi^2 \lambda_3 (\lambda_2 - 1) C}{4\lambda_2 (\tau_1 - \tau)^2}, \qquad C = \frac{1}{2}\left(\frac{1}{\delta^2} + \sum_{j=1}^{n-1} \frac{1}{(\alpha_j - \delta)^2}\right) \tag{1.23}$$

The proof of Theorem 1.3 is given in Sect. 4. In Theorems 1.1 – 1.3 the arbitrary constants γ, λ, λ_i have been chosen so as to minimize G_1; this however yields very unwieldy expressions. It can be shown that for $a_0 \to \infty$, $\tau \to 0$ and bounded μ_0, $\varepsilon_{\max}(t)$ and ε_∞ both tend to zero.

Let us now consider briefly a law of control more complex than (1.5)

$$u = kv \quad (|v| < u_0/k), \qquad u = u_0 \operatorname{sign} v \quad (|v| \geqslant u_0/k), \qquad v = \varepsilon + k_1\varepsilon'$$

Usually "the correction in velocity" is introduced in the linear closed systems in order to improve their dynamic properties. If e. g. $\tau = 0$ and the polynomial $L(p)$ has a real zero $p_1 = -a_0$ which is nearest to the imaginary axis, it is expedient to set $k_1 = a_0^{-1}$. The factor $a_0^{-1} p + 1$ appearing in the numerator of the transfer function of the open system cancels in this case with the corresponding factor appearing in $L(p)$. This enables the degree of stability of the closed system to be increased through Theorem 1.2 and the estimate of the largest accumulated error to be reduced by virtue of Theorem 1.3.

The problem of the influence of the correction in velocity" on ε_∞ when the controlling signal is restricted remains unsolved. It can however be shown that Theorem 1.1 remains valid in this case. When k_1 is chosen such that $k_1 p + 1$ is one of the factors of $L(p)$, Theorems 1.2 and 1.3 are also valid, and the system will remain linear as long as

$$v_\infty \leqslant u_0 k^{-1}, \quad v_\infty = \lim \max |v(t)| \; (t \to \infty, \; x \in F)$$

2. Estimation of the maximum error in a nonlinear system. This involves the proof of Theorem 1.1.

A. Apply the Laplace transformation to the second equation of (1.2). Taking (1.1), (1.6) into account we have

$$\mathrm{E}(p) = X(p) - Y(p) = p^{-1}\Phi(p) - (p L(p))^{-1} e^{-p\tau} U(p)$$

Employing the theorems on convolution and integration of the original function gives

$$\varepsilon(t) = \int_0^t [\varphi(t_1) - s_1(t - t_1) u(t_1)] \, dt_1, \qquad s_1(t) \fallingdotseq \frac{e^{-p\tau}}{pL(p)}, \quad \varphi(t) \fallingdotseq \Phi(p) \tag{2.1}$$

The symbol $\fallingdotseq$ denotes the correspondence between the original function and its

Laplace transform. Let

$$\varepsilon(a) = \frac{u_0}{k}, \quad \varepsilon(t) \geqslant \frac{u_0}{k} \quad (t \in [a, c]), \quad \varepsilon(b) = \max_t \varepsilon(t) \quad (t \in [a, c]) \tag{2.2}$$

From (2.1), (2.2), (1.5) it follows that

$$u(t) = u_0 \qquad (t \in [a, b])$$

$$\varepsilon(b) \leqslant m(b-a) + u_0 \left\{ \frac{1}{k} - \int_a^b s_1(b-t_1)\,dt_1 + \int_0^a |s_1(a-t_1) - s_1(b-t_1)|\,dt_1 \right\} \tag{2.3}$$

Let us obtain a lower bound for the second term and an upper bound for the third term in the braces of (2.3). From (2.1) we obtain

$$s_1(t) = 1 + q_1(t), \quad q_1(t) \risingdotseq Q_1(p) = (pL(p)\,e^{p\tau})^{-1} - p^{-1} \tag{2.4}$$

The following estimate which is valid for $q_1(t)$ will be proved in Subsection C so as to avoid interrupting the present argument

$$\begin{aligned} |q_1(t)| &< e^{-a_0\gamma t}(D_0 + D_1(e^{a_0\gamma\tau} - 1)) \qquad (t \geqslant \tau) \\ |q_1(t)| &< e^{-a_0\gamma t}(D_0 + D_1(e^{a_0\gamma t} - 1)) \qquad (0 \leqslant t \leqslant \tau) \end{aligned} \tag{2.5}$$

The quantities a_0, γ, D_0, D_1 appearing in (2.5) have been defined in Theorem 1.1. Let us assume that

$$b - a \geqslant \tau, \qquad b \geqslant \tau, \qquad a \geqslant \tau \tag{2.6}$$

Using (2.5) we obtain

$$\int_0^{b-a} s_1(t)\,dt > b - a - \frac{D_0}{a_0\gamma}(1 - e^{-a_0\gamma(b-a)}) - \tau D_1 + \frac{D_1(e^{a_0\gamma\tau} - 1)}{a_0\gamma e^{a_0\gamma(b-a)}} \tag{2.7}$$

We now turn to estimation of the third term in the braces of (2.3)

$$J_1 = \int_0^a |s_1(a-t_1) - s_1(b-t_1)|\,dt_1 = \int_{b-a}^b |s_2(z)|\,dz, \quad s_2(z) = s_1(z-b+a) - s_1(z) \tag{2.8}$$

From (2.1) it follows that

$$s_2(z) \risingdotseq G(p)\Psi(p), \quad G(p) = \frac{e^{-p\tau}}{L(p)} \risingdotseq g(t), \quad \Psi(p) = \frac{e^{-p(b-a)} - 1}{p} \risingdotseq \psi(t) \tag{2.9}$$

By the convolution theorem we have

$$s_2(z) = \int_0^z g(z - z_1)\,\psi(z_1)\,dz_1$$

From the Laplace transform of $\psi(z_1)$ follows

$$\psi(z_1) = -1 \quad (0 \leqslant z_1 \leqslant b-a), \quad \psi(z_1) = 0 \qquad (z_1 > b-a)$$

Hence

$$s_2(z) = -\int_0^{b-a} g(z - z_1)\,dz_1 \qquad (z > b-a) \tag{2.10}$$

From (2.9) we obtain

$$g(z) = l(z-\tau) \quad (z > \tau), \qquad g(z) = 0 \quad (z \leqslant \tau), \qquad l(z) \risingdotseq L^{-1}(p)$$

In Subsection B we quote the following estimate:

$$|l(t)| \leqslant D_1 a_0\gamma e^{-a_0\gamma t}$$

Using this estimate and (2.10) we obtain

$$\begin{aligned} |s_2(z)| &\leqslant D_1 e^{a_0\gamma\tau}(e^{a_0\gamma(b-a)} - 1)\,e^{-a_0\gamma z} \qquad (z \geqslant b-a+\tau) \\ |s_2(z)| &\leqslant D_1 e^{a_0\gamma\tau}(e^{-a_0\gamma\tau} - e^{-a_0\gamma z}) \qquad (b-a \leqslant z < b-a+\tau) \end{aligned} \tag{2.11}$$

Relations (2.8), (2.11) yield the estimate

$$u_0 J_1 \leqslant u_0 D_1 e^{a_0\gamma\tau} \left\{\tau e^{-a_0\gamma\tau} - \frac{e^{-a_0\gamma(b-a)} - e^{-a_0\gamma b} + e^{-a_0\gamma a} - e^{-a_0\gamma\tau}}{a_0\gamma}\right\} \qquad (2.12)$$

Let us insert the right-hand terms of (2.7), (2.12) into (2.3). Since

$$e^{-a_0\gamma b} - e^{-a_0\gamma a} \leqslant 0, \qquad b - a = \eta$$

we find that

$$\varepsilon(b) < \Phi_1 = u_0 \left\{\frac{1}{k} + \frac{D_0 + D_1}{a_0\gamma} + 2\tau D_1 + \frac{e^{-a_0\gamma\eta}}{a_0\gamma}\left[D_1 - D_0 - 2D_1 e^{a_0\gamma\tau}\right]\right\} - (u_0 - m)\eta \qquad (2.13)$$

The function $\Phi_1(\eta)$ reaches its maximum value at the point

$$\eta^* = (a_0\gamma)^{-1} \ln\left[u_0 (D_0 - D_1 + 2D_1 e^{a_0\gamma\tau})(u_0 - m)^{-1}\right]$$

Inserting η^* into (2.13) we obtain estimate (1.9) for $\varepsilon(b)$. Since the right-hand part of (1.9) is independent of c, and $\varepsilon(t)$ reaches its maximum value on the interval $[a\ c]$ at the point b, (1.9) yields the estimate for ε_∞.

In the above discussion it was assumed that conditions (2.6) are satisfied. Using similar methods for the remaining cases it will be seen that the case just studied gives the largest value for the right-hand part of (2.3).

B. Estimation of $|l(t)|$.

$$l(t) \doteqdot \frac{1}{L(p)} = \prod_{j=1}^{n-1} \frac{p_j}{p_j - p}, \qquad p_j = -\alpha_j(1 + i\mu_j)$$

From (1.8) it follows that $\operatorname{Re} p_1 = -a_0$.

Let us set the notation

$$\chi(t) = |p_1 p_2| (\alpha_2 - \alpha_1)^{-1} (e^{-\alpha_1 t} - e^{-\alpha_2 t}), \qquad l_{12}(t) \doteqdot p_1 p_2 (p - p_1)^{-1} (p - p_2)^{-1}$$

The convolution theorem yields

$$e^{a_0\gamma t} |l_{12}(t)| \leqslant \chi(t) e^{a_0\gamma t} \qquad (0<\gamma<1) \qquad (2.14)$$

Now replace the right-hand side of (2.14) by its maximum value. Taking into account (1.10) we have

$$|l_{12}| \leqslant h_1 |p_1 p_2| (\beta_1\beta_2\alpha_1\alpha_2)^{-1} e^{-a_0\gamma t}, \quad h_1 = a_0(1-\gamma)(a_0(1-\gamma)(\alpha_2 - a_0\gamma)^{-1})^{\varkappa} \quad (2.15)$$

Using this estimate and the convolution theorem we obtain the following inequality for $|l_{13}(t)|$

$$l_{13}(t) \doteqdot \prod_{j=1}^{3} \frac{p_j}{p - p_j}, \quad |l_{13}| \leqslant \frac{|p_3|}{e^{\alpha_3 t}} \int_0^t |l_{12}(t_1)| e^{\alpha_3 t_1} dt_1 \leqslant \prod_{j=1}^{3} \frac{|p_j|}{\alpha_j\beta_j} h_1 e^{-a_0\gamma t}$$

Repeating the above procedure successively we arrive at the estimate

$$|l_{1,n-1}(t)| \equiv |l(t)| \leqslant \prod_{j=1}^{n-1} \frac{|p_j|}{\alpha_j\beta_j} h_1 e^{-a_0\gamma t} = D_1 a_0 \gamma e^{-a_0\gamma t} \qquad (2.16)$$

If $\alpha_2 = a_0$, relation (1.10) is obtained by passing in (2.15) to the limit at $\alpha_2 \to a_0$. If $n = 2$, we have $l(t) \doteqdot p_1(p - p_1)^{-1}, \quad l(t) = a_0 e^{-a_0 t}, \quad D_1 = \gamma = 1$

C. Estimation of $|q(t)|$. From (2.4) it follows that

$$q_1(t) \doteqdot Q_2(p) + Q_3(p), \qquad Q_2 = (pL(p))^{-1} - p^{-1} \doteqdot q_2(t), \quad Q_3 = L^{-1}\Phi \doteqdot q_3(t) \quad (2.17)$$

$$\Phi_\tau(p) = (e^{-p\tau} - 1)p^{-1} \doteqdot \varphi_\tau(t), \quad \varphi_\tau(t) = -1 \quad (0 \leqslant t \leqslant \tau), \qquad \varphi_\tau(t) = 0 \quad (t > \tau)$$

Using the convolution theorem and taking into account (2.16), (2.17) we obtain

$$|q_3| \leqslant D_1(1 - e^{-a_0\gamma t}) \quad (0 \leqslant t \leqslant \tau), \qquad |q_3| \leqslant D_1(e^{a_0\gamma\tau} - 1)e^{-a_0\gamma t} \quad (t>\tau) \qquad (2.18)$$

Since $Q_2(p)$ has no poles to the right of the line (1.11), the inversion theorem yields

$$|q_2| \leqslant \frac{e^{-a_0\gamma t}}{2\pi} \lim_{c\to\infty} \int_{-c}^{c} \frac{\pi_1(\omega)\sqrt{a_0^2\gamma^2+\omega^2}}{(a_0^2\gamma^2+\omega^2)}\, d\omega, \quad \pi_1 = \prod_{j=1}^{n-1} \left|1 - \frac{(-a_0\gamma + i\omega)}{p_j}\right|^{-1} \tag{2.19}$$

At least one zero of $L(p)$ lies on the line (1.11). Considering the case $p_1 = -a_0$ we have

$$\sqrt{a_0^2\gamma^2+\omega^2}\,|p_1 + a_0\gamma - i\omega|^{-1} < 1 \quad (0<\gamma<{}^1/_2) \tag{2.20}$$

The coefficients of the function $\pi_1(\omega)$ corresponding to the real zeros of $L(p)$ are estimated from above by the quantity ν_j, and the coefficients corresponding to complex conjugate zeros of $L(p)$, by the quantity ν_j^2 where ν_j is defined by the formula (1.13). This can be easily verified by direct computation of the maxima of these coefficients.

If no real zeros of $L(p)$ lie on the line (1.11) then at least one complex conjugate pair of zeros p_1 and p_2 lie on this line and we have

$$\sqrt{a_0^2\gamma^2+\omega^2}\,(|p_1 + a_0\gamma - i\omega|\,|p_2 + a_0\gamma - i\omega|)^{-1} \leqslant r \quad (0<\gamma<(1+\sqrt{2})^{-1}) \tag{2.21}$$

The value of r is given in (1.13) and is obtained by calculating the maximum value of the left-hand side of (2.21). Inserting the estimates for the coefficients of $\pi_1(\omega)$ and (2.20) or (2.21) into (2.19) yields

$$|q_2(t)| \leqslant D_0 e^{-a_0\gamma t}, \quad (1)\quad D_0 = \frac{1}{2\gamma}\prod_{j=2}^{n-1}\nu_j, \qquad (2)\quad D_0 = \frac{r}{2\gamma}\prod_{j=3}^{n-1}\nu_j \tag{2.22}$$

Adding the inequalities (2.18) and (2.22) yields the estimate (2.5) for $|q_1(t)|$.

3. Estimation of the degree of stability of linear systems with lag. This involves a proof of Theorem 1.2.

The argument principle [9] implies that the necessary and sufficient condition for all zeros of the quasipolynomial $N(p)$ to lie on the left of the straight line $\mathrm{Re}\,p = -\delta$ is given by the following equation for the increment of the argument of the function $N(p)$:

$$\Delta \arg N_\delta(\omega) = {}^1/_2 n\pi \qquad (0\leqslant\omega<\infty), \qquad N_\delta(\omega) \equiv N(-\delta + i\omega) \tag{3.1}$$

Suppose that a value $\omega_1 > 0$ has been found for some $\delta \in (0, a_0)$ such that the amplification factor k satisfies the condition (1.16) of Theorem 1.2

$$\begin{gathered}\mathrm{Im}\, N_\delta(\omega) \geqslant 0 \quad (0\leqslant\omega\leqslant\omega_1), \quad \mathrm{Im}\, Q_\delta(\omega) \geqslant 0 \quad (0\leqslant\omega\leqslant\omega_1) \\ |Q_\delta(\omega)| \geqslant \lambda_1 |Q_\delta(0)| \quad (\omega>\omega_1), \qquad Q_\delta(\omega) \equiv Q(-\delta+i\omega)\end{gathered} \tag{3.2}$$

Under these conditions the equality (3.1) holds and the quantity δ will be the lower bound of the degree of stability δ^* of the quasipolynomial $N(p)$. Indeed, we find that

$$\Delta \arg N_\delta(\omega) = \Delta \arg Q_\delta(\omega) + \Delta \arg N_\delta(\omega)\, Q_\delta^{-1}(\omega) \tag{3.3}$$

holds on any interval of variation of ω.

Since the degree of stability a_0 of the polynomial $L(p)$ is greater than δ and $Q = pL(p)$, then $n-1$ zeros of $Q(p)$ lie to the left of the line $\mathrm{Re}\,p = -\delta$ and the remaining one $p = 0$, to the right. Therefore

$$Q_\delta(0) < 0, \quad \Delta \arg Q_\delta(\omega) = {}^1/_2(n-2)\pi \quad (0\leqslant\omega<\infty) \tag{3.4}$$

By condition (1.16) of the theorem $N_\delta(0) > 0$, consequently taking into account (3.2) we find that the point

$$D = N_\delta(\omega)\, Q_\delta^{-1}(\omega) = 1 + ke^{-(-\delta+i\omega)\tau}\, Q_\delta^{-1}(\omega)$$

lies in the lower semiplane for any $\omega > \omega_1$. In addition, by (3.2) and (1.16), point D lies in the right semiplane and tends to the point (1, 0) on the real axis as $\omega \to \infty$. This implies that

$$\Delta \arg N_\delta(\omega) Q_\delta^{-1}(\omega) = \pi \qquad (0 \leqslant \omega < \infty) \tag{3.5}$$

Equations (3.3) – (3.5) yield (3.1).

To obtain the quantity δ for which conditions (3.2) hold, we denote

$$Q_\delta(\omega) = \rho_\delta(\omega) e^{i\varphi_\delta(\omega)}, \quad k = \lambda\rho_\delta(0)e^{-\delta\tau} \quad (1 \leqslant \lambda \leqslant \lambda_1) \tag{3.6}$$

The first condition of (3.2) is equivalent to the inequality

$$\rho_\delta(\omega) \sin \varphi_\delta(\omega) > \lambda\rho_\delta(0) \sin \omega\tau \tag{3.7}$$

Since

$$\delta \in (0, a_0), \qquad Q(p) = pL(p)$$

we have

$$\varphi_\delta = -\pi - \operatorname{arc\,tg} \omega\delta^{-1} + \psi_\delta, \quad \psi_\delta = \arg L(-\delta + i\omega), \quad \psi_\delta(0) = 0 \tag{3.8}$$

It can easily be verified that the inequality

$$\psi_\delta'(\omega) \leqslant \sum_{j=1}^{n-1} \frac{1}{\alpha_j - \delta} \leqslant \frac{n-1}{a_0 - \delta} \tag{3.9}$$

holds for any value of ω

If the condition

$$0 < \delta\gamma_1(\delta) < 1, \quad \gamma_1(\delta) = \tau_1 + (n-1)\ (a_0 - \delta)^{-1}$$
$$\tau_1 = \pi\tau\lambda_1 / 2 + \lambda_0 a_0^{-1} \tag{3.10}$$

where λ_0 is an arbitrary nonnegative number, holds, then from (3.8), (3.9) we have that

$$\sin \varphi_\delta > \sin \tau_1\omega > 0 \qquad (0 \leqslant \omega \leqslant \omega_1 = \gamma_1^{-1}(\delta) \operatorname{arc\,cos} \delta\gamma_1(\delta) < \pi / 2\tau_1) \tag{3.11}$$

i.e. the second condition of (3.2) is fulfilled. Since

$$\sin \tau_1\omega > \frac{2\omega\tau_1}{\pi} \ (\omega \in [0, \omega_1]), \qquad \lambda \in [1, \lambda_1], \quad \frac{\sin x}{x} < 1 \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right)$$

the inequality (3.7) is fulfilled automatically, provided that

$$\rho_\delta(\omega)\, \rho_\delta^{-1}(0) > 1, \qquad \omega \in [0, \omega_1] \tag{3.12}$$

Let us denote

$$\rho_\delta(\omega)\, \rho_\delta^{-1}(0) = \Gamma_1(z) H_1^{-1}(z) = I_1(z), \qquad z = \omega^2 \tag{3.13}$$

In this equation $\Gamma_1(z)$ combines all the coefficients corresponding to the real zeros of $Q(p)$, and $H_1^{-1}(z)$ combines the coefficients corresponding to the complex zeros of $Q(p)$. Discarding the terms which contain z in higher than the first order in the polynomial $\Gamma_1(z)$ with positive coefficients, we obtain

$$\Gamma_1(z) > \Gamma(z) = 1 + kz^*, \qquad k^* = \frac{1}{\delta^2} + \sum{}^* \frac{1}{(\alpha_j - \delta)^2} \tag{3.14}$$

where the sum contains the terms corresponding to the real zeros of $L(p)$.

Every coefficient $h_j(z)$ of the function $H_1(z)$ corresponding to a pair of complex conjugate zeros of $L(p)$ is bounded from above by the function

$$b_j(z) = 1 + k_j^2 z \qquad (0 \leqslant z \leqslant z_j = (A_j^2 - 1) k_j^{-2})$$
$$b_j(z) = A_j^2 \ (z > z_j) \tag{3.15}$$

If the curve $h_j(z)$ has a maximum, then the horizontal part of the polygonal line $b_j(z)$ touches $h_j(z)$ at its maximum while its inclined part not only touches $h_j(z)$ but has with it another common point $z = 0$. If $h_j(z)$ decreases monotonously, $b_j(z)$ is parallel to the abscissa. It can be verified that the coefficients k_j and A_j decrease with decreasing δ. From (3.10) it follows that $\delta < \delta_1$, the latter being the smallest root of (1.18). For this reason we set $\delta = \delta_1$ in the expression for k_j and A_j

$$k_j^2 = 0 \quad (0 \leqslant \mu_j \leqslant y_j) \qquad k_j^2 = 0.25\ \alpha_j^{-2} y_j^{-2}, \quad (\mu_j \geqslant \sqrt{3} y_j)$$
$$k_j^2 = 2\,(\mu_j^2 - y_j^2)(\mu_j^2 + y_j^2)^{-2} \qquad (y_j < \mu_j < \sqrt{3} y_j)$$

The quantities y_j and A_j are defined by (1.18). Replacing the coefficients $h_j(z)$ with their upper bounds given by (3.15) yields an upper bound $H_2(z)$ for $H_1(z)$. If the derivatives $b'(z)$ are replaced in the expression for $H_2'(z)$ by their maximum k_j^2, we obtain

$$H_1(z) < H(z), \quad H(z) = 1 + k_0 z \quad (z < z_0 = (W^2 - 1)\, k_0^{-1})$$

$$H(z) = W^2 \quad (z > z_0), \qquad k_0 = W^2 \prod_{j=1}^{n-1} \frac{k_j}{A_j} \tag{3.16}$$

Taking into account (3.14) and (3.16) we find that (3.13) yields

$$I_1(z) > \Gamma(z) H^{-1}(z) = I(z) \geqslant (1 + k^* z) / W^2 \tag{3.17}$$

Let a value of δ be chosen such that the inequalities

$$I'(z) \geqslant 0 \quad (z \geqslant 0), \qquad I(z_1) > \lambda_1^2, \quad z_1 = \omega_1^2 \tag{3.18}$$

hold. Then

$$I(z) \geqslant \lambda_1^2 \ (z \geqslant z_1) \tag{3.19}$$

It can easily be verified that

$$k^* > k_0 \qquad (\delta < \delta_1) \tag{3.20}$$

The latter condition is sufficient for the first inequality of (3.18) to hold, and this implies that (3.12) also holds. From (3.16) and (3.11) it follows that the second inequality of (3.18) holds, provided that

$$\gamma_1^{-1}(\delta)\ \arccos\ (\delta\gamma_1(\delta)) \geqslant (\lambda_1^2 W^2 - 1)^{1/2}\ (k^*)^{-1/2} \tag{3.21}$$

Since the functions

$$\gamma_1^{-1}(\delta) \qquad (0 \leqslant \delta \leqslant \delta_1), \qquad \arccos x \quad (0 \leqslant x \leqslant 1)$$
$$\delta_1 \gamma_1(\delta_1) = 1, \quad k^* > \delta^{-2}$$

are convex, the inequality (3.21) is automatically satisfied, provided that

$$0.5\ \pi\gamma_1^{-1}(0)\ (1 - \delta\,\delta_1^{-1}) \geqslant \delta\sqrt{\lambda_1^2 W^2 - 1} \tag{3.22}$$

The latter expression yields the quantity δ defined by (1.17) and this quantity represents the upper limit of the degree of stability of $N(p)$. Thus all the conditions of (3.2) are satisfied. The first condition holds since it is equivalent to (3.7) which follows from (3.11), (3.12), (3.17) and (3.18), the second condition applying by virtue of (3.11) and the third condition following from (3.17) and (3.19).

Replacing the inequality (3.9) by

$$\psi_\delta'(\omega) \leqslant \sum_{j=1}^{n-1} \frac{1}{\alpha_j - \delta} \leqslant \frac{q}{a_0 - \delta} + \sum \frac{1}{\alpha_j - a_0}$$

yields a corollary to Theorem 1.2. In the above expression q denotes the number of zeros of $L(p)$ lying on the straight line (1.11) and the sum is taken over all the remaining zeros of $L(p)$.

4. Estimation of maximum error in a linear system with lag. Let us prove Theorem 1.3. We apply the Laplace transformation to Eq. (1.7). Taking (1.1), (1.2), (1.6) and (1.15) into account gives

$$\mathrm{E}(p) = G(p)\,\Phi(p), \quad G(p) = L(p)\,N^{-1}(p), \quad \mathrm{E}(p) \risingdotseq \varepsilon(t), \quad \Phi(p) \risingdotseq \varphi(t)$$

Since by Theorem 1.2 the degree of stability of $N(p)$ exceeds δ, $G(p)$ is a transform and the growth index of its original $g(t)$ is smaller than δ (see [10]). From the convolution theorem (1.11) it follows that

$$\varepsilon(t) = \int_0^t g(\tau)\,\varphi(t-\tau)\,d\tau, \quad \varepsilon_{\max}(t) = m\int_0^t |g(\tau)|\,d\tau \tag{4.1}$$

Let us obtain an estimate for $|g(t)|$ using the inversion theorem and integrating along the line $p = -\delta + i\omega$. Setting

$$G(p) = p^{-1} + A(p), \quad A(p) = -kN^{-1}(p)\,p^{-1}\,e^{-p\tau} \tag{4.2}$$

we find that

$$g(t) = \lim_{c\to\infty} \int_{-\delta-ic}^{-\delta+ic} \frac{G(p)\,e^{pt}}{2\pi}\,dp = \frac{e^{-\delta t}}{\pi}\lim_{c\to\infty}\int_0^c A(-\delta + i\omega)\,e^{i\omega t}\,d\omega \tag{4.3}$$

since the integral of the first term of $G(p)$ is equal to zero.

Let us estimate the second integral in (4.3). We set

$$z = \omega^2, \quad A_\delta(z) = |A(-\delta + i\omega)|, \quad \mu(z) = \varphi_\delta(\omega) + \omega\tau \tag{4.4}$$

Taking into account (3.6), (3.13) we find

$$A_\delta(z) = ((\delta^2 + z)\,R(z))^{-1/2}, \quad R(z) = \left(\frac{I_1(z)}{\lambda} - 1\right)^2 + \frac{4}{\lambda}\,I_1(z)\cos^2\frac{\mu(z)}{2} \tag{4.5}$$

Using (3.11) and inequality

$$\sin\left(\sqrt{z}\,\frac{(\tau_1-\tau)}{2}\right) > \frac{\sqrt{2}}{\pi}(\tau_1-\tau)\sqrt{z} \qquad \left(0 \leqslant z \leqslant z_1 \leqslant \frac{\pi^2}{4(\tau_1-\tau)^2}\right)$$

we obtain

$$\cos^2(\mu(z)/2) > 2\pi^{-2}(\tau_1-\tau)^2 z \quad (0 \leqslant z \leqslant z_1) \tag{4.6}$$

To find the lower limit for the first term of $R(z)$ in (4.5), we make use of the fact that for any $z > 0$

$$I_1'(z) \leqslant I_1(z)\,\frac{1}{2}\left[\frac{1}{\delta^2} + \sum_{j=1}^{n-1}\frac{1}{(\alpha_j-\delta)^2}\right] = I_1(z)\,C \tag{4.7}$$

Since $1 < \lambda_2 < \lambda_1$, taking into account (3.17) and (3.18) we find

$$1 \leqslant I_1(z) < \lambda_2 \quad (0 \leqslant z < z_2 < z_1), \qquad I_1(z_2) = \lambda_2 \tag{4.8}$$

From (4.7), (4.8) and the condition $\lambda > \lambda_2$ we have

$$\begin{gathered} I_1(z) \leqslant 1 + \lambda_2 C z \qquad (0 \leqslant z \leqslant z_2) \\ (1 - I_1(z)\lambda^{-1})^2 > (1 - \lambda_2^{-1} - Cz)^2 \\ (0 \leqslant z \leqslant z_3 = (1 - \lambda_2^{-1})\;C^{-1} < z_2 < z_1) \end{gathered} \tag{4.9}$$

Inequalities (4.6) and (4.9) hold on $[0, z_3]$ simultaneously. By (4.5) and the condition $\lambda < \lambda_3$ we have, on this interval,

$$R\ (z) > (1 - \lambda_2^{-1} - Cz)^2 + 8\pi^{-2}\lambda_3^{-1}(\tau_1 - \tau)^2 z \equiv R_1\ (z) \equiv \equiv (B - Cz)^2 + Dz \qquad (4.10)$$

Let us denote by R_0^2 the minimum of $R_1\ (z)$ for $z \geqslant 0$. We can easily verify that

$$R_0^2 = B^2\left(D_3 = \frac{2BC}{D} < 1\right), \quad R_0^2 = \frac{2B^2}{D_3} - \frac{B^2}{D_3^2}\ (D_3 > 1),$$

$$R\ (z) > \frac{BD}{C} > R_0^2 \qquad (z_3 \leqslant z \leqslant z_1) \qquad (4.11)$$

hence

$$R\ (z) \geqslant R_0^2 \qquad (z \in [0,\ z_1]) \qquad (4.12)$$

Let us now find the estimate for $A(z)$. From (4.12), (3.17), (3.18) and the inequalities

$$k^* > \delta^{-2}, \qquad \lambda \leqslant \lambda_3 < \lambda_1$$

follows

$$A > R_0^{-1}\ (\delta^2 + z)^{-1/2}\ (0 \leqslant z \leqslant z_1), \qquad A > (\delta^2 + z)^{-1/2}((1 + k^*z)^{1/2} \times \times (\lambda_3 W)^{-1} - 1)^{-1} > (\delta^2 + z)^{-1}\ (\delta^2 + z_1)^{1/2}\ ((1 + k^*z_1)^{1/2}(\lambda_3 W)^{-1} - 1)^{-1}$$

$$(z_1 = \omega_1^2 < z = \omega^2 < \infty) \qquad (4.13)$$

Replacing the integrand expression in (4.3) by its modulus and using (4.13) we obtain the upper limit for $|\ g\ (t)\ |$, which on insertion into (4.1) yields the estimates (1.21) and (1.22) for $\varepsilon_{max}\ (t)$ and ε_∞.

BIBLIOGRAPHY

1. Bulgakov, B. V., Accumulation of perturbations in linear oscillating systems with constant parameters. Dokl. Akad. Nauk SSSR, Vol. 51, №5, 1946.
2. Bulgakov, B. V. and Kuzovkov, N. T., On the accumulation of disturbances in linear systems with variable parameters. PMM Vol. 14, №1, 1950.
3. Gnoenskii, L. S., On the relation of certain quality indices in linear, steady-state control systems. Dokl. Akad. Nauk SSSR, Vol. 181, №1, 1968.
4. Gnoenskii, L. S., Kamenskii, G. A. and El'sgol'ts, L. E., Mathematical Principles of the Theory of Control Systems. M. "Nauka", 1969.
5. Bendrikov, G. A. and Teodorchik, K. F., Trajectories of the Roots of Linear Automatic Systems. M. "Nauka", 1964.
6. Bendrikov, G. A. and Konev, F. B., Application of the method of root trajectories to the study of linear control systems with "pure" lag (free parameter τ). Vest. MGU, Ser. 3. Fiz., astron., №3, 1968.
7. Tsypkin, Ia. Z., Upper bound of the degree of stability of one-contour automatic control systems. Avtomat. i telemekhan., Vol. 13, №4, 1952.
8. Gnoenskii, L. S. and Kamenskii, G. A., Effects of lag in computers on the quality indices of the control systems. Proc. of the Seminar on the theory of differential equations with a deviating argument. Vol. 7, M., Univ. Druzhby narodov im. Patrisa Lumumby, 1969.
9. Voronov, A. A., Fundamentals of the Automatic Control Theory. Pt. 2. M.–L., "Energiia", 1966.
10. Pinney, E., Ordinary Differential-Difference Equations. Berkeley, University of California Press, 1958

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